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DYNAMICS OF AN ELASTIC SATELLITE—II*

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3. SIGNIFICANCE OF GRAVITATIONAL EFFECTS

To provide perspective and to establish motivation for the stability analysis of a deformable space vehicle in a torque-free state, the significance of gravitational effects, i.e. the importance of the terms in equations (2.87)–(2.98) containing F^G_{1j} and T^G_{ij}, is assessed by examining the magnitudes of these quantities for satellites rotating at rates of spin that are moderate or high in comparison with the mean motion n.

When a "spin factor" α is defined as (see (2.86) for $\bar{\omega}$)

$$\alpha = \frac{\bar{\omega}}{n} \tag{3.1}$$

and the normalizing quantity $\bar{\omega}$ is taken as the initial spin rate of the satellite in reference frame N, α reflects the number of revolutions the satellite makes in N during each orbit. The quantity $n^2\zeta^{-3}$, which appears in each of F^G_{1j} and T^G_{ij} (see equations (2.63)–(2.65) and (2.116)), may be expressed in terms of the spin factor α as

$$n^2\zeta^{-3} = \left(\frac{\bar{\omega}}{\alpha}\right)^2\zeta^{-3}. \tag{3.2}$$

Consequently, the gravitational terms appear in the normalized equations (2.87)–(2.95) as

$$\left.\begin{aligned} \frac{F^G_{11}}{mL\bar{\omega}^2} &= -(1{\cdot}5\zeta^{-3}\alpha^{-2})(\mathrm{c}^2\psi_2\mathrm{s}\psi_3\mathrm{c}\psi_3) \\ \frac{F^G_{12}}{mL\bar{\omega}^2} &= -(0{\cdot}5\zeta^{-3}\alpha^{-2})(1-3\mathrm{c}^2\psi_2\mathrm{s}^2\psi_3) \\ \frac{F^G_{13}}{mL\bar{\omega}^2} &= -(1{\cdot}5\zeta^{-3}\alpha^{-2})(\mathrm{c}\psi_2\mathrm{s}\psi_2\mathrm{s}\psi_3). \end{aligned}\right\} \tag{3.3}$$

* Part I of this paper appeared in the previous issue of this Journal, and Part III will appear in the next. All references are listed in Part I.

$$\left.\begin{aligned}
\frac{T^G_{01}}{A\bar{\omega}^2} &= (3k_1\zeta^{-3}\alpha^{-2})\left[\left(1-\frac{5}{2}\frac{L}{a}\zeta^{-1}c\psi_2 s\psi_3\right)(c\psi_2 s\psi_2 s\psi_3)+\frac{\zeta^{-1}}{2}\frac{L}{a}s\psi_2\right]\\
\frac{T^G_{02}}{B\bar{\omega}^2} &= (3k_2\zeta^{-3}\alpha^{-2})\left[\left(1-\frac{5}{2}\frac{L}{a}\zeta^{-1}c\psi_2 s\psi_3\right)(-c\psi_2 s\psi_2 c\psi_3)\right]\\
\frac{T^G_{03}}{C\bar{\omega}^2} &= (3k_3\zeta^{-3}\alpha^{-2})\left[\left(1-\frac{5}{2}\frac{L}{a}\zeta^{-1}c\psi_2 s\psi_3\right)(c^2\psi_2 c\psi_3 s\psi_3)+\frac{\zeta^{-1}}{2}\frac{L}{a}c\psi_2 c\psi_3\right].
\end{aligned}\right\} \quad (3.4)$$

$$\left.\begin{aligned}
\frac{T^G_{11}}{A\bar{\omega}^2} &= (3k_1\zeta^{-3}\alpha^{-2}) \quad \text{[see (2.65)]}\\
\frac{T^G_{12}}{B\bar{\omega}^2} &= (3k_2\zeta^{-3}\alpha^{-2}) \quad \text{[see (2.65)]}\\
\frac{T^G_{13}}{C\bar{\omega}^2} &= (3k_3\zeta^{-3}\alpha^{-2}) \quad \text{[see (2.65)]}.
\end{aligned}\right\} \quad (3.5)$$

Equations (3.3)–(3.5) show that the influence of the gravitational quantities depends primarily on the spin factor and on orbit eccentricity. As the maximum absolute values of ζ^{-1} and k_j are $(1-\varepsilon)^{-1}$ and 1, respectively, no gravitational quantity in (3.3)–(3.5) can exceed $3(1-\varepsilon)^{-3}\alpha^{-2}$, regardless of satellite orientation. Thus, when the orbit eccentricity ε becomes small and the spin factor α becomes large, the importance of the normalized gravitational terms is diminished. For example, if $\alpha = 1000$ and $\varepsilon = 0{\cdot}1$, the maximum gravitational quantity in equations (3.3)–(3.5) has a magnitude less than $4{\cdot}2\times10^{-6}$. As the other terms of the differential equations (2.87)–(2.95) are not similarly small, it appears that under these circumstances the gravitational forces have relatively little effect on the motion of the satellite. Moreover, it is the size of α and not $\bar{\omega}$ which determines the relative importance of gravitational effects. This may be seen by combining equations (2.107) and (3.1), which yields

$$\alpha = \frac{\bar{\omega}T}{2\pi}. \quad (3.6)$$

Hence, if the period is sufficiently large, $\bar{\omega}$ can be small while α is large. This is of interest in connection with space station design because, according to NASA preliminary environmental studies [1]* on rotating manned space stations, $\bar{\omega}$ is likely to be small, e.g. in the neighborhood of 3 rev/min. However, even with such a small $\bar{\omega}$, α would be 900 for a 300 min period. By way of contrast, the Earth could be considered to have an α and $\bar{\omega}$ of approximately 365 and 1/1440 rev/min, respectively.

4. INSTABILITY

Equations of motion for torque-free state

We now consider the stability of a deformable space vehicle in a torque-free state. The governing differential equations of motion, obtained by dropping F^G_{1j} and T^G_{ij} from equations (2.87)–(2.95), will hereafter be referred to as equations† (2.87)′–(2.95)′.

* Numbers in brackets designate references listed in the bibliography at the end of the first part of this paper.
† See footnote on p. 347 of the previous issue.

Once all gravitational terms have been eliminated, it is not necessary to specify either the orientation in N of the spin axis or the path of P_*, because equations (2.87)′–(2.95)′ are uncoupled from the attitude equations (2.96)–(2.98) and the orbital equations (2.114) and (2.115). That is, the vehicle motion no longer reflects the vehicle orientation in N or the path followed by the vehicle mass center.

A word of caution is appropriate before we proceed with the stability analysis. Although the gravitational quantities which were dropped from equations (2.87)–(2.95) were shown to be small, there is no absolute assurance that the stability predictions made without them will agree with the stability judgements made with them; for the nature of the solutions of differential equations may on occasion be altered by neglecting even small terms. (For such an example, see Sokolnikoff and Redheffer [20].) Since the stability analysis which follows is based entirely on equations (2.87)′–(2.85)′, the stability question for a vehicle in orbit may require re-examination. For any particular case, this re-examination might well be carried out by integrating numerically the full, nonlinear equations of motions (equations (2.87)–(2.98), (2.114) and (2.115)).

Constant solution of the equations of motion

Intuitively, it seems possible that the elastic space vehicle under consideration can have an initial spin about an axis parallel to X_3^0 and X_3^1 and that the resulting motion is then a steady spin of (constant) rate $\bar{\omega}$ accompanied by a constant elastic extension $\bar{p}_2$. Indeed, this turns out to be the case. If certain restrictions are placed on the connecting structure, i.e. on the stiffness matrix, the above motion, called "simple spin", satisfies the differential equations exactly.

The aforementioned restriction on the structure is that

$$S_{k2} = S_{2k} = 0, \qquad k = 1, 3, 4, 5, 6\,(\neq 2). \tag{4.1}$$

In other words, an induced elongation of amount p_2 must not be accompanied by a translation p_1 or p_3 or by a rotation θ_1, θ_2, or θ_3; and conversely, a displacement of amount p_1 or p_3 or a rotation of amount θ_1, θ_2, or θ_3 must not be accompanied by a translation p_2.

The constant solution of equations (2.87)′–(2.95)′ corresponding to simple spin is

$$\left.\begin{array}{lll} p_1 = 0 & p_2 = \bar{p}_2 & p_3 = 0 \\ \omega_1 = 0 & \omega_2 = 0 & \omega_3 = \bar{\omega} \\ \theta_1 = 0 & \theta_2 = 0 & \theta_3 = 0 \end{array}\right\} \tag{4.2}$$

where barred quantities indicate constants. With the exception of equation (2.88)′, all terms in equations (2.87)′–(2.95)′ are identically equal to zero when substitutions from equation (4.2) are made. From equations (2.88)′, (2.29), (4.1), and (4.2), one can see that the constant extension $\bar{p}_2$, for a constant spin rate $\bar{\omega}$ must be

$$\bar{p}_2 = \frac{L}{2S_{22}/m\bar{\omega}^2 - 1}\,. \tag{4.3}$$

Instability of constant solution

In order to study the stability of the simple spin motion, the differential equations of motion (equations (2.87)′–(2.95)′) are linearized around the constant solution, equations

(4.2), by taking

$$\left.\begin{array}{lll} p_1 = p_1^* & p_2 = \bar{p}_2 + p_2^* & p_3 = p_3^* \\ \omega_1 = \omega_1^* & \omega_2 = \omega_2^* & \omega_3 = \bar{\omega} + \omega_3^* \\ \theta_1 = \theta_1^* & \theta_2 = \theta_2^* & \theta_3 = \theta_3^* \end{array}\right\} \tag{4.4}$$

where the starred quantities are unknown functions of time. When this transformation is performed and all terms that are nonlinear in starred quantities are dropped, the resulting equations are

$$\begin{aligned}(p_1^*/L)'' - (p_1^*/L) - 2(p_2^*/L)' + [(2/mL\bar{\omega}^2) + (L/C\bar{\omega}^2)]\\ \times [S_{11}p_1^* + S_{13}p_3^* + S_{14}\theta_1^* + S_{15}\theta_2^* + S_{16}\theta_3^*] - (1/C\bar{\omega}^2)\\ \times (S_{61}p_1^* + S_{63}p_3^* + S_{64}\theta_1^* + S_{65}\theta_2^* + S_{66}\theta_3^*) = 0\end{aligned} \tag{4.5}$$

$$(p_2^*/L)'' - (p_2^*/L) + 2(p_1^*/L)' - 2(\omega_3^*/\bar{\omega})(1 + \bar{p}_2/L) + (2/mL\bar{\omega}^2)(S_{22}p_2^*) = 0$$

(4.3) (4.6)

$$\begin{aligned}(p_3^*/L)'' + (1 + k_1)(1 + \bar{p}_2/L)\omega_2^*/\bar{\omega} + [(2/mL\bar{\omega}^2) + (L/A\bar{\omega}^2)]\\ \times [S_{31}p_1^* + S_{33}p_3^* + S_{34}\theta_1^* + S_{35}\theta_2^* + S_{36}\theta_3^*]\\ + (1/A\bar{\omega}^2)(S_{41}p_1^* + S_{43}p_3^* + S_{44}\theta_1^* + S_{45}\theta_2^* + S_{46}\theta_3^*) = 0\end{aligned} \tag{4.7}$$

$$\begin{aligned}(\omega_1^*/\bar{\omega})' - k_1\omega_2^*/\bar{\omega} - (1/A\bar{\omega}^2)[(S_{41}p_1^* + S_{43}p_3^* + S_{44}\theta_1^* + S_{45}\theta_2^* + S_{46}\theta_3^*)\\ + L(S_{31}p_1^* + S_{33}p_3^* + S_{34}\theta_1^* + S_{35}\theta_2^* + S_{36}\theta_3^*)] = 0\end{aligned} \tag{4.8}$$

$$(\omega_2^*/\bar{\omega})' - k_2\omega_1^*/\bar{\omega} - (1/B\bar{\omega}^2)(S_{51}p_1^* + S_{53}p_3^* + S_{54}\theta_1^* + S_{55}\theta_2^* + S_{56}\theta_3^*) = 0 \tag{4.9}$$

$$\begin{aligned}(\omega_3^*/\bar{\omega})' - (1/C\bar{\omega}^2)[(S_{61}p_1^* + S_{63}p_3^* + S_{64}\theta_1^* + S_{65}\theta_2^* + S_{66}\theta_3^*)\\ - L(S_{11}p_1^* + S_{13}p_3^* + S_{14}\theta_1^* + S_{15}\theta_2^* + S_{16}\theta_3^*)] = 0\end{aligned} \tag{4.10}$$

$$\begin{aligned}(\theta_1^*)'' - k_1\theta_1^* - (1 + k_1)(\theta_2^*)' + (1/A\bar{\omega}^2)[2(S_{41}p_1^* + S_{43}p_3^* + S_{44}\theta_1^* + S_{45}\theta_2^* + S_{46}\theta_3^*)\\ + L(S_{31}p_1^* + S_{33}p_3^* + S_{34}\theta_1^* + S_{35}\theta_2^* + S_{36}\theta_3^*)] = 0\end{aligned} \tag{4.11}$$

$$(\theta_2^*)'' + k_2\theta_2^* + (1 - k_2)(\theta_1^*)' + (2/B\bar{\omega}^2)(S_{51}p_1^* + S_{53}p_3^* + S_{54}\theta_1^* + S_{55}\theta_2^* + S_{56}\theta_3^*) = 0 \tag{4.12}$$

$$\begin{aligned}(\theta_3^*)'' + (1/C\bar{\omega}^2)[2(S_{61}p_1^* + S_{63}p_3^* + S_{64}\theta_1^* + S_{65}\theta_2^* + S_{66}\theta_3^*)\\ - L(S_{11}p_1^* + S_{13}p_3^* + S_{14}\theta_1^* + S_{15}\theta_2^* + S_{16}\theta_3^*) = 0.\end{aligned} \tag{4.13}$$

As equations (4.5)–(4.13) are ordinary, linear, homogeneous differential equations with constant coefficients, the instability (in the Lyapunov sense) of their zero solution may be studied by examining the roots of the characteristic equation. Specifically, if any of the roots has a positive real part, the zero solution is unstable (see Cesari [21], p. 19).

An instability prediction based on an analysis of the linearized equations holds for the zero solution of the corresponding nonlinear equations (see Cesari [21], pp. 92–93). Conversely, a stability prediction based on linearized equations does not hold for the zero solution of the corresponding nonlinear equations unless all characteristic roots have negative real parts (asymptotic stability). Thus, only instability predictions can be expected in the present problem, because asymptotic stability is ruled out by the fact that no energy dissipation mechanism has been incorporated in the system.

The characteristic polynomial for equations (4.5)–(4.13) can be conveniently presented in the form of a determinant Δ defined as

$$\Delta = |\Delta_{ij}| \tag{4.14}$$

with elements

$$\Delta_{11} = \lambda^2 - 1 + 2\frac{S_{11}}{m\bar{\omega}^2} + \frac{L}{C\bar{\omega}^2}(LS_{11} - S_{61}); \qquad \Delta_{12} = -2\lambda$$

$$\Delta_{13} = 2\frac{S_{13}}{m\bar{\omega}^2} + \frac{L}{C\bar{\omega}^2}(LS_{13} - S_{63}); \qquad \Delta_{14} = 0$$

$$\Delta_{15} = 0; \qquad \Delta_{16} = 0$$

$$\Delta_{17} = 2\frac{S_{14}}{mL\bar{\omega}^2} + \frac{1}{C\bar{\omega}^2}(LS_{14} - S_{64}); \qquad \Delta_{18} = 2\frac{S_{15}}{mL\bar{\omega}^2} + \frac{1}{C\bar{\omega}^2}(LS_{15} - S_{65})$$

$$\Delta_{19} = 2\frac{S_{16}}{mL\bar{\omega}^2} + \frac{1}{C\bar{\omega}^2}(LS_{16} - S_{66}); \qquad \Delta_{21} = 2\lambda$$

$$\Delta_{22} = \lambda^2 - 1 + 2\frac{S_{22}}{m\bar{\omega}^2}; \qquad \Delta_{23} = 0$$

$$\Delta_{24} = 0; \qquad \Delta_{25} = 0$$

$$\Delta_{26} = -2(1 + \bar{p}_2/L); \qquad \Delta_{27} = 0$$

$$\Delta_{28} = 0; \qquad \Delta_{29} = 0$$

$$\Delta_{31} = 2\frac{S_{31}}{m\bar{\omega}^2} + \left(\frac{L}{A\bar{\omega}^2}\right)(LS_{31} + S_{41}); \qquad \Delta_{32} = 0$$

$$\Delta_{33} = \lambda^2 + 2\frac{S_{33}}{m\bar{\omega}^2} + \frac{L}{A\bar{\omega}^2}(LS_{33} + S_{43}); \qquad \Delta_{34} = 0$$

$$\Delta_{35} = (1 + k_1)\left(1 + \frac{\bar{p}_2}{L}\right); \qquad \Delta_{36} = 0$$

$$\Delta_{37} = 2\frac{S_{34}}{mL\bar{\omega}^2} + \frac{1}{A\bar{\omega}^2}(LS_{34} + S_{44}); \qquad \Delta_{38} = 2\frac{S_{35}}{mL\bar{\omega}^2} + \frac{1}{A\bar{\omega}^2}(LS_{35} + S_{45})$$

$$\Delta_{39} = 2\frac{S_{36}}{mL\bar{\omega}^2} + \frac{1}{A\bar{\omega}^2}(LS_{36} + S_{46}); \qquad \Delta_{41} = -\frac{L}{A\bar{\omega}^2}(LS_{31} + S_{41})$$

$$\Delta_{42} = 0; \qquad \Delta_{43} = -\frac{L}{A\bar{\omega}^2}(LS_{33} + S_{43})$$

$$\Delta_{44} = \lambda; \qquad \Delta_{45} = -k_1$$

$$\Delta_{46} = 0; \qquad \Delta_{47} = -\frac{1}{A\bar{\omega}^2}(LS_{34} + S_{44})$$

$$\Delta_{48} = -\frac{1}{A\varpi^2}(LS_{35}+S_{45}); \qquad \Delta_{49} = -\frac{1}{A\varpi^2}(LS_{36}+S_{46})$$

$$\Delta_{51} = -L\frac{S_{51}}{B\varpi^2}; \qquad \Delta_{52} = 0$$

$$\Delta_{53}\, M - L\frac{S_{53}}{B\varpi^2}; \qquad \Delta_{54} = -k_2$$

$$\Delta_{55} = \lambda; \qquad \Delta_{56} = 0$$

$$\Delta_{57} = -\frac{S_{54}}{B\varpi^2}; \qquad \Delta_{58} = -\frac{S_{55}}{B\varpi^2}$$

$$\Delta_{59} = -\frac{S_{56}}{B\varpi^2}; \qquad \Delta_{61} = \frac{L}{C\varpi^2}(LS_{11}-S_{61})$$

$$\Delta_{62} = 0; \qquad \Delta_{63} = \frac{L}{C\varpi^2}(LS_{13}-S_{63})$$

$$\Delta_{64} = 0; \qquad \Delta_{65} = 0$$

$$\Delta_{66} = \lambda; \qquad \Delta_{67} = \frac{1}{C\varpi^2}(LS_{14}-S_{64})$$

$$\Delta_{68} = \frac{1}{C\varpi^2}(LS_{15}-S_{65}); \qquad \Delta_{69} = \frac{1}{C\varpi^2}(LS_{16}-S_{66})$$

$$\Delta_{71} = \frac{L}{A\varpi^2}(2S_{41}+LS_{31}); \qquad \Delta_{72} = 0$$

$$\Delta_{73} = \frac{L}{A\varpi^2}(2S_{43}+LS_{33}); \qquad \Delta_{74} = 0$$

$$\Delta_{75} = 0; \qquad \Delta_{76} = 0$$

$$\Delta_{77} = \lambda^2 - k_1 + \frac{1}{A\varpi^2}(2S_{44}+LS_{34}); \qquad \Delta_{78} = -(1+k_2)\lambda + \frac{1}{A\varpi^2}(2S_{45}+LS_{35})$$

$$\Delta_{79} = \frac{1}{A\varpi^2}(2S_{46}+LS_{36}); \qquad \Delta_{81} = 2L\frac{S_{51}}{B\varpi^2}$$

$$\Delta_{82} = 0; \qquad \Delta_{83} = 2L\frac{S_{53}}{B\varpi^2}$$

$$\Delta_{84} = 0; \qquad \Delta_{85} = 0$$

$$\Delta_{86} = 0; \qquad \Delta_{87} = (1-k_2)\lambda + 2\frac{S_{54}}{B\varpi^2}$$

$$\Delta_{88} = \lambda^2 + k_2 + 2\frac{S_{55}}{B\varpi^2}; \qquad \Delta_{89} = 2\frac{S_{56}}{B\varpi^2}$$

$$\Delta_{91} = \frac{L}{C\varpi^2}(2S_{61}-LS_{11}); \qquad \Delta_{92} = 0$$

$$\Delta_{93} = \frac{L}{C\varpi^2}(2S_{63}-LS_{13}); \qquad \Delta_{94} = 0$$

$$\Delta_{95} = 0; \qquad \Delta_{96} = 0$$

$$\Delta_{97} = \frac{1}{C\varpi^2}(2S_{64}-LS_{14}); \qquad \Delta_{98} = \frac{1}{C\varpi^2}(2S_{65}-LS_{15})$$

$$\Delta_{99} = \lambda^2+\frac{1}{C\varpi^2}(2S_{66}-LS_{16}).$$

The characteristic polynomials can be simplified considerably by setting equal to zero all elements of the stiffness matrix except for S_{11}, S_{22}, S_{33}, S_{44}, S_{55}, S_{66}, S_{16}, S_{61}, S_{34}, and S_{43} and, in addition, imposing the requirements

$$\left.\begin{aligned} 2S_{16}-LS_{11} &\underset{(2.31)}{=} 2S_{61}-LS_{11} = 0 \\ 2S_{34}+LS_{33} &\underset{(2.31)}{=} 2S_{43}+LS_{33} = 0. \end{aligned}\right\} \tag{4.15}$$

These restrictions on the connecting structure are not so prohibitive as might at first appear. With a modest amount of structural symmetry, these requirements on $[S]$ can easily be satisfied. Two examples which meet these restrictions are discussed in the sequel, one being a circular shaft, the other a truss.

Expansion of the characteristic polynomial Δ is facilitated by the introduction of the following ten dimensionless parameters:

$$\left.\begin{aligned} \delta_1 &= k_1 = (B-C)/A & \delta_6 &= S_{11}/m\varpi^2 \\ \delta_2 &= k_2 = (C-A)/B & \delta_7 &= S_{22}/m\varpi^2 \\ \delta_3 &= \quad = S_{44}/A\varpi^2 & \delta_8 &= S_{33}/m\varpi^2 \\ \delta_4 &= \quad = S_{55}/B\varpi^2 & \delta_9 &= L^2S_{33}/A\varpi^2 \\ \delta_5 &= \quad = S_{66}/C\varpi^2 & \delta_{10} &= L^2S_{11}/C\varpi^2. \end{aligned}\right\} \tag{4.16}$$

As may be verified by application of the Buckingham Pi Theorem [22], at most nine of the ten parameters in equations (4.16) can be independent of each other; and indeed, δ_{10} is connected to other parameters through the relationship

$$\delta_{10} = \frac{\delta_6\delta_9}{\delta_8(1-\delta_1)}\left[1-\frac{\delta_1+\delta_2}{1+\delta_1\delta_2}\right]. \tag{4.17}$$

After rearrangement of rows and columns of Δ, one can express the determinant Δ as

$$\Delta \underset{(4.14)}{=} \begin{bmatrix} \Delta_{66} & \Delta_{61} & 0 & 0 & 0 & 0 & 0 & 0 & \Delta_{69} \\ 0 & \Delta_{11} & \Delta_{12} & 0 & 0 & 0 & 0 & 0 & \Delta_{19} \\ \Delta_{26} & \Delta_{21} & \Delta_{22} & 0 & 0 & 0 & 0 & 0 & 0 \\ 0 & 0 & 0 & \Delta_{44} & \Delta_{45} & \Delta_{43} & \Delta_{47} & 0 & 0 \\ 0 & 0 & 0 & \Delta_{54} & \Delta_{55} & 0 & 0 & \Delta_{58} & 0 \\ 0 & 0 & 0 & 0 & \Delta_{35} & \Delta_{33} & \Delta_{37} & 0 & 0 \\ 0 & 0 & 0 & 0 & 0 & 0 & \Delta_{77} & \Delta_{78} & 0 \\ 0 & 0 & 0 & 0 & 0 & 0 & \Delta_{87} & \Delta_{88} & 0 \\ 0 & 0 & 0 & 0 & 0 & 0 & 0 & 0 & \Delta_{99} \end{bmatrix} \tag{4.18}$$

where the nonzero elements are (see definition of Δ_{ij} (4.14)–(4.16))

$$\begin{aligned}
\Delta_{11} &= \lambda^2-1+2\delta_6+\delta_{10}/2; & \Delta_{12} &= -2\lambda \\
\Delta_{19} &= \delta_6-\delta_5+\delta_{10}/2; & \Delta_{21} &= 2\lambda \\
\Delta_{22} &= \lambda^2-1+2\delta_7; & \Delta_{26} &= -2(1+\bar{p}_2/L) \\
\Delta_{33} &= \lambda^2+2\delta_8+\delta_9/2; & \Delta_{35} &= (1+\delta_1)(1+\bar{p}_2/L) \\
\Delta_{37} &= -\delta_8+\delta_3-\delta_9/2; & \Delta_{43} &= -\delta_9/2 \\
\Delta_{44} &= \lambda; & \Delta_{45} &= -\delta_1 \\
\Delta_{47} &= -\delta_3+\delta_9/2; & \Delta_{54} &= -\delta_2 \\
\Delta_{55} &= \lambda; & \Delta_{58} &= -\delta_4 \\
\Delta_{61} &= \delta_{10}/2; & \Delta_{66} &= \lambda \\
\Delta_{69} &= -\delta_5+\delta_{10}/2; & \Delta_{77} &= \lambda^2-\delta_1+2\delta_3-\delta_9/2 \\
\Delta_{78} &= -(1+\delta_1)\lambda; & \Delta_{87} &= (1-\delta_2)\lambda \\
\Delta_{88} &= \lambda^2+\delta_2+2\delta_4; & \Delta_{99} &= \lambda^2+2\delta_5-\delta_{10}/2
\end{aligned}$$

with δ_{10} and $\bar{p}_2/L$ used as abbreviations. δ_{10} is given in equation (4.17) and $\bar{p}_2/L$, expressed in terms of independent parameters, is

$$\frac{\bar{p}_2}{L} = \frac{1}{2\delta_7-1}. \tag{4.19}$$

The determinant Δ, expressed in the quasi-triangular form of equation (4.18), may be expanded in such a way that the associated characteristic equation can be presented in factored form. This is accomplished by using Laplace's theorem (see Shilov [23]) for determinant expansion, which here leads to

$$\lambda\{\lambda^4+\lambda^2(2\delta_6+2\delta_7+\delta_{10}/2+2)+(1-2\delta_7)(1-2\delta_6-\delta_{10}/2)+2\delta_{10}(1+\bar{p}_2/L)\}$$
$$\times\{\lambda^4+\lambda^2(2\delta_8+\delta_9/2-\delta_1\delta_2)-\delta_1\delta_2(2\delta_8+\delta_9/2)+\delta_2\delta_9(1+\delta_1)(1+\bar{p}_2/L)/2\}$$
$$\times\{\lambda^4+\lambda^2(2\delta_3+2\delta_4-\delta_9/2-\delta_1\delta_2+1)+(\delta_2+2\delta_4)(2\delta_3-\delta_1-\delta_9/2)\}$$
$$\times\{\lambda^2+2\delta_5-\delta_{10}/2\} = 0. \quad (4.20)$$

Instability inequalities

The fifteenth degree equation (4.20) is seen to have one linear, one quadratic, and three biquadratic factors. Because at least one root of a quadratic equation of the form

$$\lambda^2+c = 0$$

has a positive real part if

$$c < 0$$

and a biquadratic equation of the form

$$\lambda^4+b\lambda^2+c = 0$$

must have at least one root with a positive real part if

$$b < 0 \quad \text{or} \quad c < 0 \quad \text{or} \quad b^2-4c < 0$$

the following "instability inequalities" can be constructed by reference to equation (4.20):

$$2\delta_6+2\delta_7+\delta_{10}/2+2 < 0 \quad (4.21)$$

$$(1-2\delta_7)(1-2\delta_6-\delta_{10}/2)+2\delta_{10}(1+\bar{p}_2/L) < 0 \quad (4.22)$$

$$[2(1+\delta_6+\delta_7)+\delta_{10}/2]^2-4[(1-2\delta_7)(1-2\delta_6-\delta_{10}/2)+2\delta_{10}(1+\bar{p}_2/L)] < 0 \quad (4.23)$$

$$2\delta_8+\delta_9/2-\delta_1\delta_2 < 0 \quad (4.24)$$

$$\delta_2[-2\delta_1\delta_8+\delta_9/2+\delta_9(1+\delta_1)\bar{p}_2/2L] < 0 \quad (4.25)$$

$$(2\delta_8+\delta_9/2-\delta_1\delta_2)^2-4\delta_2[-2\delta_1\delta_8+\delta_9/2+\delta_9(1+\delta_1)\bar{p}_2/2L] < 0 \quad (4.26)$$

$$2\delta_3+2\delta_4-\delta_9/2-\delta_1\delta_2+1 < 0 \quad (4.27)$$

$$(\delta_2+2\delta_4)(2\delta_3-\delta_1-\delta_9/2) < 0 \quad (4.28)$$

$$[2\delta_3+2\delta_4-\delta_9/2-\delta_1\delta_2+1]^2-4(\delta_2+2\delta_4)(2\delta_3-\delta_1-\delta_9/2) < 0 \quad (4.29)$$

$$2\delta_5-\delta_{10}/2 < 0. \quad (4.30)$$

Examination of (4.21) shows that this inequality can never be satisfied because

$$\delta_3,\ldots,\delta_{10} \geq 0. \quad (4.31)$$

To verify this, observe that the elements on the principal diagonal of the stiffness matrix are non-negative quantities, i.e.

$$S_{kk} \geq 0, \qquad k = 1,\ldots,6 \quad (4.32)$$

and recall that the other quantities A, B, C, m, L^2, and $\bar{\omega}^2$ appearing in equation (4.16) are intrinsically positive.

Now, nine instability inequalities in (4.22)–(4.30) in terms of parameters representing inertia, geometric, and elastic properties as well as the spin rate of the vehicle, i.e. $\delta_1, \ldots, \delta_9$, can be used to predict instabilities of the elastic space vehicle. As will be shown later, there exist two basic types of instabilities. The first type may be regarded as an "attitude instability," because it is associated with the fact that ω_1 and ω_2 cannot be kept arbitrarily small by sufficiently restricting the initial departure from simple spin. The second type, characterized by unbounded growth of p_j or θ_j, and called "deformation instability," occurs when the space vehicle fails to maintain basic geometric integrity after simple spin has been disturbed.

In order to make instability predictions for a particular space vehicle configuration, one may proceed as follows:

(1) Given A, B, C, m, L, $\bar{\omega}$, and $[S]$,
(2) verify that $[S]$ is of proper form by checking equation (4.15);
(3) determine $\delta_1, \ldots, \delta_{10}$ from equation (4.16), and $\bar{p}_2/L$ with equation (4.19);
(4) make substitutions into (4.22)–(4.30). Then,
(5) if any one of the inequalities is satisfied, attitude and/or deformation instability is assured.

Inequality interpretation

It is both interesting and informative to relate instabilities of the elastic space vehicle to instabilities of the associated rigid body. Preliminary to a review of instabilities of a rigid body in a torque-free state, it is recalled that I_1, I_2, and I_3 represent the centroidal principal moments of inertia of the associated rigid body R_* for P_* (see equations (2.46) and (2.47)). The fact that the motion of a rigid body whose angular velocity is parallel to a principal axis, say axis X_3, is stable if I_3 is the greatest or least moment of inertia, and is unstable if I_3 is an intermediate moment of inertia has traditionally been presented as a consequence of Euler's dynamical equations. (See Routh [24] for a discussion of this topic.) The introduction of associated rigid body inertia parameters

$$K_1 = (I_2 - I_3)/I_1, \qquad K_2 = (I_3 - I_1)/I_2 \tag{4.33}$$

where, for any real body,

$$|K_1| < 1, \qquad |K_2| < 1 \tag{4.34}$$

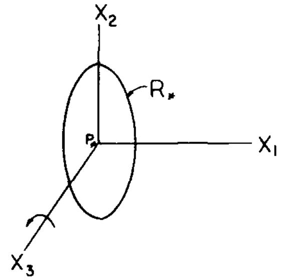
makes it possible to present these facts in a simple graphical form, as shown in Fig. 7.

To establish a relationship between the associated rigid body parameters (K_1, K_2) and the space vehicle parameters $(\delta_1, \ldots, \delta_9)$, an auxiliary parameter ξ is defined as

$$\xi = \frac{1}{4A/mL^2 + 1} \underset{(4.16)}{=} \frac{1}{4\delta_8/\delta_9 + 1}. \tag{4.35}$$

Note that

$$0 < \xi < 1. \tag{4.36}$$



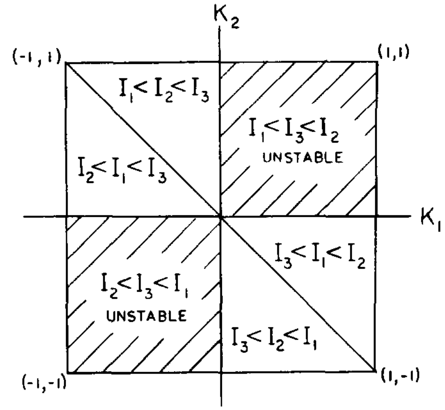


FIG. 7. Associated rigid body instability chart.

Now, with substitutions from equations (2.47), (4.33) and (4.35) into equation (2.99),

$$\delta_1 = k_1 = (K_1+\xi)/(1-\xi) \tag{4.37}$$

$$\delta_2 = k_2 = K_2 \tag{4.38}$$

where it is noted that for any real end bodies R_0 and R_1

$$|k_1| < 1, \qquad |k_2| < 1. \tag{4.39}$$

The nine vehicle parameters are now $K_1, K_2, \delta_3, \ldots, \delta_8$, and ξ instead of $\delta_1, \ldots, \delta_9$; and the instability inequalities can be transformed accordingly. In addition to equations (4.37) and (4.38), which are needed to transform δ_1 and δ_2, the parameters δ_9 and δ_{10} in terms of this new set of vehicle parameters are

$$\delta_9 = \frac{4\delta_8\xi}{1-\xi} \tag{4.40}$$

$$\delta_{10} = \frac{4\delta_6\xi}{1-K_1-2\xi}\left[\frac{1-K_1-K_2+K_1K_2+2K_2\xi-2\xi}{1+K_1K_2+K_2\xi-\xi}\right]. \tag{4.41}$$

Due to the complexity of equation (4.41), not all instability inequalities are here transformed. However, as a demonstration of what may be done to interpret (4.22)–(4.30), the inequalities (4.24), (4.25), and (4.28) are transformed and related to the K_1–K_2 plane for the associated rigid body.

Because (4.25) is particularly interesting after transformation, it is considered first. After substitution from equations (4.35), (4.37), (4.38), and (4.40) into (4.25), the inequality becomes

$$K_2\left(\frac{2\delta_8}{1-\xi}\right)\left[-K_1+\xi\left(\frac{1+K_1}{1-\xi}\right)\frac{\bar{p}_2}{L}\right] < 0. \tag{4.42}$$

As $2\delta_8/(1-\xi)$ is a positive quantity, inequality (4.42) holds either if

$$K_1 > \frac{\bar{p}_2}{L}\frac{\xi}{1-\xi-\xi(\bar{p}_2/L)} \quad \text{and} \quad K_2 > 0 \tag{4.43}$$

or if

$$K_1 < \frac{\bar{p}_2}{L}\frac{\xi}{1-\xi-\xi(\bar{p}_2/L)} \quad \text{and} \quad K_2 < 0 \tag{4.44}$$

where $\bar{p}_2$ (see equation (4.19)) is the elastic elongation associated with simple spin. A

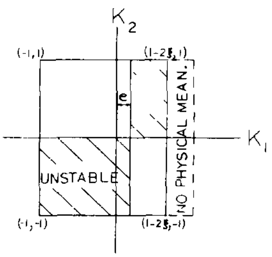


FIG. 8. Elastic space vehicle instability chart.

graphical representation of (4.43) and (4.44) is shown in Fig. 8, where e is defined as

$$e = \frac{\bar{p}_2}{L}\frac{\xi}{1-\xi-\xi(\bar{p}_2/L)} \tag{4.45}$$

and where for real systems the K_1–K_2 plane is restricted by the inequalities

$$-1 < K_1 < 1-2\xi, \qquad -1 < K_2 < 1 \tag{4.46}$$

which are deduced from (4.37), (4.38) and (4.39). It can now be seen that, when the connecting structure is made stiffer, i.e. $\bar{p}_2 \to 0$, the instability chart shown in Fig. 8 approaches that of the associated rigid body, Fig. 7. Note that there are portions of the K_1–K_2 plane which have opposite classifications in Figs. 7 and 8, which means that elastic elongation can be regarded as a stabilizing as well as a destabilizing factor.

When transformed, inequalities (4.24) and (4.28) yield

$$(K_1+\xi)K_2 > 2\delta_8 \tag{4.47}$$

and

$$(K_2+2\delta_4)[-K_1-\xi(2\delta_3+2\delta_8+1)+2\delta_3] < 0 \tag{4.48}$$

respectively. When these inequalities are represented on the K_1–K_2 plane, the result appears as shown in Fig. 9 where δ_3, δ_4, δ_8, and ξ are free parameters. The boundaries of the unstable regions, governed by the inequalities (4.47) and (4.48), may shift on the K_1–K_2 plane according to the selected values of the free parameters. Thus, these boundaries may be shifted either outward, so as to be outside the physically meaningful portion of the parameter space, or they may be shifted inward by changing the elastic properties of the connection, the mass of R_i, and/or the spin rate of the vehicle.

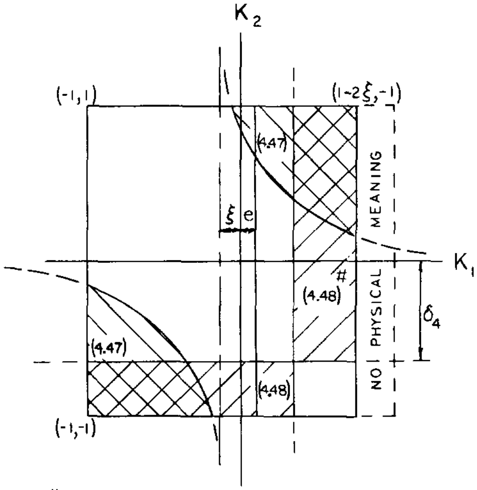


FIG. 9. Instability chart.

In a similar fashion, the other inequalities may be transformed and curves drawn on the K_1–K_2 plane; however, enough has been done to establish that a stability prediction based on a rigid body analysis may be contrary to the prediction made when the elastic properties of the system are taken into account. In fact, as may be seen by examining inequalities (4.43), (4.44), (4.47), (4.48), and Figs. 8 and 9, the instability boundaries cutting across the physically meaningful portion of the K_1–K_2 plane could include much of this parameter space and thus lead to incorrect stability predictions if the connecting structure, the inertia characteristics of the end bodies, and the spin rate of the space vehicle were chosen without regard to the instability inequalities (4.22)–(4.30).

Interpretations of the instability inequalities (4.22)–(4.30), other than those obtained by relating them to the associated rigid body K_1–K_2 plane, can be made and may prove useful. For example, (4.27) and (4.30) can never be satisfied if

$$4S_{44}-L^2S_{33} > 0 \quad \text{and} \quad 4S_{66}-L^2S_{11} > 0 \,. \tag{4.49}$$

Thus, assuming that the other inequalities (4.22)–(4.26), (4.28), and (4.29) are not satisfied, instabilities may be avoided by requiring that the connecting structure have a stiffness matrix which satisfies the inequalities in (4.49).

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